POISSON COHOMOLOGY OF BROKEN LEFSCHETZ FIBRATIONS

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Abstract. We compute the formal Poisson cohomology of a broken Lefschetz fibration by calculating it at fold singularities and singular points. Near a fold singularity the computation reduces to that for a point singularity in 3 dimensions. For the Poisson cohomology around singular points we adapt techniques developed for the Sklyanin algebra. As a side result, we give compact formulas for the Poisson coboundary operator of an arbitrary Jacobian Poisson structure in 4 dimensions.

1. Introduction

Poisson geometry originated in the Hamiltonian formalism of classical mechanics, and plays an important role in the passage to quantum mechanics. A primary tool in studying invariants of Poisson manifolds is Poisson cohomology. It reveals important features of the geometry of a Poisson manifold such as the modular class, obstructions to deformations, normal forms and deformation quantization. However, calculating Poisson cohomology can be very difficult. The cohomology spaces are generally infinite-dimensional, and they are unknown for many Poisson structures as there is no general method for their computation. One has better chances when restricting to formal coefficients for the cochain complex of multivector fields.

For weight homogeneous Poisson algebras in 3 variables, Pichereau [19] computed (co)homology with formal coefficients under the assumption that the structure is determined by a weight homogeneous polynomial with isolated singularity i.e finite Milnor number. Following this work, Pelap [18] gave formulas for the formal (co)homology of the Sklyanin algebra. The structure there is unimodular, weight homogeneous, with two weight homogeneous Casimirs forming a complete intersection with isolated singularity. Also in dimension 4, Hong and Xu computed the Poisson cohomology of del Pezzo surfaces [10]. Under some conditions, Monnier [17] computed the formal Poisson cohomology of quadratic structures.

Broken Lefschetz fibrations (bLfs) originated as a generalization of Lefschetz pencils [6, 2], and in recent years they have found diverse applications in low-dimensional topology, symplectic geometry, and singularity theory [5, 8, 11, 21]. A bLf is a map from a 4–manifold $X$ to the 2-sphere, with a singularity set consisting of a finite collection of circles, called fold singularities, and a finite set of isolated points, also known as Lefschetz singularities. Here we determine the formal Poisson cohomology of a Poisson structure associated to a bLf.

As shown in [7], on a bLf there is an associated Poisson structure $\pi$ whose degeneracy locus coincides with the singularity set of the fibration. Due to the existence of bLfs on 4–manifolds, such a Poisson structure $\pi$ exists on any homotopy class of


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maps from $X$ to $S^2$, thus making their Poisson cohomology an interesting feature in terms of different classification questions. Although the model of $\pi$ around fold singularities is linear and around Lefschetz singularities is quadratic, one cannot use Lie algebra cohomology and [17] respectively; the Lie algebra structure is not semisimple and the quadratic one has vanishing modular vector field.

We show that the formal Poisson cohomology and its generators around folds can be computed viewing the singular circles as point singularities of a 3-manifold. In particular, the proof of Proposition 6.2 shows that one can decompose the coboundary operator in a way that isolates a point singularity in a 3-manifold and then adds an extra dimension in both the manifold and the singularity to get the fold singularity of a bLf. Restricting the Poisson structure on this 3-manifold, we get a Poisson structure determined by a weight homogeneous polynomial with isolated singularity as in [19]. The weight homogenous polynomial that we use is one of the Casimirs of the original Poisson structure around the singular circle. This property is reflected in the cohomology spaces as well, as the cohomology class of the vector field tangent to the circle is the generator of the first Poisson cohomology group.

For the Poisson cohomology around Lefschetz points we use a different strategy. Most of the 4-dimensional calculus for homology developed in [18] can be used here with some amendments related to the Poisson structure around Lefschetz point. In order to build compact formulas for the coboundary operator one needs a specific Clifford rotation $D$ of $\mathbb{R}^4$ that fixes the singularity, and an endomorphism $K$ of $\mathfrak{so}(4)$ that fixes $D$. We use these more generally at first, to compute formulas for the Poisson coboundary operator for any Jacobian Poisson structure in 4 dimensions. Then we restrict back to the Poisson cohomology of bLfs. Due to unimodularity and seen as free Casimir modules, the cohomology spaces have the same rank as the Poisson homology spaces of the Sklyanin algebra. We compute their generators explicitly as shown in the main result of the paper.

**Theorem 1.1.** Let $f: M \to S^2$ be a broken Lefschetz fibration on an oriented, smooth, closed 4-manifold $M$. Denote by $\pi \in \mathcal{X}^2(M)$ the associated Poisson structure vanishing on a finite collection of circles $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ and a finite collection of points $C = \{p_1, \ldots, p_r\}$. The formal Poisson cohomology of $(M, \pi)$ on the tubular neighbourhood $U_\Gamma$ is determined by the following free Cas- modules

$$H^0_{\pi}(U_\Gamma, \pi) \cong \mathbb{R}^m$$
$$H^1_{\pi}(U_\Gamma, \pi) \cong \mathbb{R}^m \cong \bigoplus_{i=1}^m \frac{\partial}{\partial \theta_i}$$
$$H^2_{\pi}(U_\Gamma, \pi) \cong 0$$
$$H^3_{\pi}(U_\Gamma, \pi) \cong \mathbb{R}^m \cong \bigoplus_{i=1}^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$$
$$H^4_{\pi}(U_\Gamma, \pi) \cong \mathbb{R}^m \cong \bigoplus_{i=1}^m \text{vol}_i$$

where

- $\text{Cas}$ denotes the algebra $\mathbb{R}[Q^1_i, Q^2_i]$ of Casimirs with $Q^1_i = \theta_i, Q^2_i = -x^2_1 + x^2_2 + x^2_3$ the representatives around each $\gamma_i$,
- $\theta_i$ is the parameter of each circle $\gamma_i$ in $\Gamma$ with normal coordinates $(x_1, x_2, x_3)$,
- $\text{vol}_i$ is the volume form $d\theta \wedge dx_1 \wedge dx_2 \wedge dx_3$ around each $\gamma_i$. 

and around \( C \) the formal Poisson cohomology is determined by the following free Cas- modules

\[
\begin{align*}
H^0_\pi(U, \pi) &\cong \mathbb{R}^r \\
H^1_\pi(U, \pi) &\cong \mathbb{R}^r \cong \bigoplus_{i=1}^r E_i \\
H^2_\pi(U, \pi) &\cong \mathbb{R}^{6r} \cong \bigoplus_{i=1}^r \left( \bigoplus_{k=1}^5 K^{-1}(\nabla \nu_k \times \nabla P_1^l) \oplus K^{-1}(\nabla P_1^l \times \nabla P_2^l) \right) \\
H^3_\pi(U, \pi) &\cong \mathbb{R}^{13r} \cong \bigoplus_{i=1}^r \left( \bigoplus_{k=1}^5 D(\nabla \nu_k) \oplus \bigoplus_{k=0}^5 \nu_k D(\nabla P_1^l) \oplus D(\nabla P_2^l) \oplus \nu_k D(\nabla P_1^l) + x_1 x_2 D(\nabla P_1^l) \right) \\
H^4_\pi(U, \pi) &\cong \mathbb{R}^{7r} \cong \bigoplus_{i=1}^r \text{span}(1, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6)
\end{align*}
\]

where

- \( \text{Cas} \) denotes the algebra of Casimirs with \( P_1^l = x_1^2 - x_2^2 + x_3^2 - x_4^2, \) \( P_2^l = 2(x_1 x_2 + x_3 x_4) \) the representatives around \( p_l \in C, \)
- \( E_i \) is the Euler vector field in coordinates \( x_1, x_2, x_3, x_4, \) around each \( p_l \)
- \((\nu_i)_{0 \leq i \leq 6} = (1, x_1, x_2, x_3, x_4, x_1 x_2, x_3 x_4).\)

The Poisson structure on a bLf, together with the Poisson structure on near-symplectic 4-manifolds studied in [3] and log-symplectic structures on 4-manifolds, are examples of singular Poisson structures for any possible combination of degeneracies in the rank of a Poisson structure. The Poisson bivector associated to a bLF is of rank 2 or 0, whereas on a 4-manifold log-symplectic structures are Poisson structures of rank 4 or 2, and near-symplectic manifolds have a Poisson bivector of rank 4 or 0. Hence, together with the Poisson cohomology of near-symplectic manifolds [3] and the Poisson cohomology of log-symplectic manifolds [9, 16], here we complete the Poisson cohomology computation for these structures.

The paper is structured as follows. In Section 2.1 we recall some basic notions of Poisson geometry and cohomology and in Section 2.2 we describe the operators that we will use in the computation of Poisson cohomology around Lefschetz points. The latter is presented in section 5. Before that, in section 4 and proposition 4.1 we give general and compact formulas for the Poisson coboundary operators of Jacobian Poisson structures. Section 6 and proposition 6.2 concludes the paper with the computation of the formal Poisson cohomology around fold singularities.

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2. Preliminaries

2.1. Basic definitions and unimodular Poisson structures. We first recall some basic objects from Poisson geometry, see [12] for details. A Poisson structure on a smooth manifold $M$ is a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the Leibniz rule $\{fg,h\} = f\{g,h\} + g\{f,h\}$. Equivalently, a Poisson bivector field $\pi \in \Gamma(\wedge^2 TM)$ is a bivector field satisfying $[\pi,\pi]_{SN} = 0$ for the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN} : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M)$ and where $\mathfrak{X}^k(M) = \Gamma(\Lambda^k TM)$. The Poisson bracket and bivector field are mutually determined by $\{f,g\} = \langle \pi, df \otimes dg \rangle$.

A bivector field $\pi$ induces an operator $d_\pi : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ by $d_\pi(X) = [\pi,X]_{SN}$, and if $\pi$ is Poisson then $d_\pi^2 = 0$. The pair $(\mathfrak{X}(M),d_\pi)$ is called the Lichnerowicz-Poisson cochain complex, and $H^k_\pi(M) := \ker (d_\pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)) / \text{im} (d_\pi : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M))$, $k \in \mathbb{N}_0$ are the Poisson cohomology spaces of $(M,\pi)$. The zeroth cohomology space $H^0_\pi(M)$ contains the Casimir functions, that is $f \in C^\infty(M)$ such that $\{f,g\} = 0$, $\forall g \in C^\infty(M)$. The first cohomology group $H^1_\pi(M)$ is the quotient of Poisson modulo Hamiltonian vector fields, i.e $X \in \mathfrak{X}(M)$ satisfying $\mathcal{L}_X \pi = 0$ modulo $X \in \mathfrak{X}(M)$ such that $X = d_\pi(f)$. Furthermore, $H^2_\pi(M)$ is the quotient of infinitesimal deformations of $\pi$ modulo trivial deformations and $H^3_\pi(M)$ contains the obstructions to formal deformations of $\pi$.

Contraction with $\pi$ defines a vector bundle homomorphism $\pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$, usually referred to as the anchor map. Pointwise it is written as $\pi^\sharp_p(\alpha_p) = \pi_p(\alpha_p,\cdot)$ and can be extended to a $C^\infty(M)$-linear homomorphism

$$\wedge^\bullet \pi^\sharp : \Omega^\bullet(M) \to \mathfrak{X}^\bullet(M),$$

which we denote again by $\pi^\sharp$. The Hamiltonian vector field of $f \in C^\infty(M)$ is then $X_f = \pi^\sharp(df)$.

Consider an orientable Poisson manifold with positive oriented volume form $\Omega$. The vector field $Y^\Omega : C^\infty(M) \to C^\infty(M)$ defined by

$$\mathcal{L}_{X_f} \Omega = (Y^\Omega f) \Omega,$$

is a Poisson vector field known as the modular vector field with respect to $\Omega$. One can check directly that there is a canonically defined Poisson cohomology class $[Y^\Omega]$ called the modular class of $(M,\pi)$. If $[Y^\Omega] = 0$ then $(M,\pi)$ is called unimodular.

Let $\star$ denote the family of $C^\infty$-linear operators $\star : \mathfrak{X}^k(M) \to \Omega^{n-k}(M)$ defined by $\star X := \iota_X \Omega$. When $(M,\pi)$ is unimodular, $\star$ induces an isomorphism between the $k$-th Poisson cohomology group $H^k_\pi(M)$ and the $(n-k)$-th Poisson homology group $H^\pi_{n-k}(M)$.

2.2. Some operators. We henceforth restrict in dimension $n = 4$. Set $\partial_i : \mathfrak{X}^1 \to \mathfrak{X}^1$, $C = C^\infty(\mathbb{R}^4)$, $\mathfrak{X}^k = \mathfrak{X}^k(\mathbb{R}^4)$ and identify $k$—vector fields with the ordered tuples of their coefficient functions as follows

$$\iota_1 : \mathfrak{X}^1 \xrightarrow{\approx} \mathbb{R}^4, \quad X = \sum_{i=1}^4 f_i \partial_i \mapsto (f_1, f_2, f_3, f_4)^T$$
$$\iota_2 : \mathbb{R}^2 \xrightarrow{\cong} \mathbb{C}^6, \quad Y = \sum_{i<j=1}^{4} f_{ij}\partial_{ij} \mapsto (f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34})^T$$

$$\iota_3 : \mathbb{R}^3 \xrightarrow{\cong} \mathbb{C}^4, \quad W = \sum_{i<j<l=1}^{4} f_{ijk}\partial_{ijk} \mapsto (f_{123}, f_{124}, f_{134}, f_{234})^T$$

and

$$\iota_0 : \mathbb{C}^4 \xrightarrow{\cong} \mathbb{R}^0, \quad \iota_4 : \mathbb{R}^4 \xrightarrow{\cong} \mathbb{C}, \quad \iota_0(f) = f, \quad \iota_4(f\partial_{1234}) = f.$$
$C^\infty$-isomorphisms $D_1 : \mathfrak{X}^1 \to \mathfrak{X}^1$ such that $\sum_{i=1}^4 f_i \partial_i \mapsto \sum_{i=1}^4 g_i \partial_i$ and $C^\infty$-isomorphisms $D_3 : \mathfrak{X}^3 \to \mathfrak{X}^3$ with

$$\sum_{i=1}^4 f_i \partial_{123} + f_2 \partial_{124} + f_3 \partial_{134} + f_4 \partial_{234} \mapsto \sum_{i=1}^4 g_i \partial_{123} + g_2 \partial_{124} + g_3 \partial_{134} + g_4 \partial_{234}.$$  

Let $\mathcal{I} : \mathfrak{X}^1 \to \mathfrak{X}^3$ be the $C^\infty$-isomorphism induced by $D$. Then $D_3$ can be equivalently determined by the equation

$$\mathcal{I} = D_3 \circ \star^{-1} \circ I,$$

and $D_1$ by the equation

$$D_1 = I^{-1} \circ \star \circ \mathcal{I}.$$  

For the rest of the paper, we fix $D_3 = D_3$ and $D_1 = D_1$ to be the isomorphisms determined by the choice $D = \text{Id}_{C^4}$. Since the context will be clear, we will denote them both as $D$. In matrix form,

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$  

**Remark 2.1.** One can check that as an element of $SO(4)$, $D$ represents a right-isoclinic rotation of the 4-dimensional space. Since it has purely imaginary adjoint eigenvalues of multiplicity 2, it is a Clifford rotation. Such rotations do not have a fixed plane, however they do have a fixed point. In our use of $D$ in section 5, this point is identified with the Lefschetz singularity.

Consider now automorphisms $K : C^6 \to C^6$ with

$$(f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}) \mapsto (g_{12}, g_{13}, g_{14}, g_{23}, g_{24}, g_{34}).$$

Via (3) there is a one-to-one correspondence with isomorphisms $K : \mathfrak{X}^2 \to \mathfrak{X}^2$ with

$$\sum_{1=i<j=4} f_{ij} \partial_{ij} \mapsto \sum_{1=i<j=4} g_{ij} \partial_{ij}.$$  

Fix $K = K : \mathfrak{X}^2 \to \mathfrak{X}^2$ to be the isomorphism satisfying the equation

$$I^{-1}(f(I(K(X)))) = K(I^{-1}(\star(X))).$$

In matrix form,

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  

**Remark 2.2.** The operator $K$ can be further viewed as an element of $SO(6)$ and more generally as a linear transformation of $\mathbb{R}^6$. Equip the latter the standard bilinear form and the norm it induces. Given the bijective isometry between this normed space and $\mathfrak{so}(4)$ equipped with the Pfaffian $\text{Pf}(A) = A_{12}A_{34} + A_{31}A_{24} + A_{23}A_{14}$, $K$ defines an automorphism $\mathfrak{R} : \mathfrak{so}(4) \to \mathfrak{so}(4)$. Considering $D$ as an element of $\mathfrak{so}(4)$, it is $\mathfrak{R}(D) = -iD$. 
Finally denote by $\bar{\cdot}$ the operator $\star: C^4 \times C^6 \to C^4$, $(X, Y) \mapsto -\ast (X \wedge Y)$. In matrix form, for $X = [f_1, f_2, f_3]$, $Y = [g_1, g_2, g_3, g_4, g_5, g_6]^T$, it is

$$X \bar{\otimes} Y = \begin{bmatrix}
-f_4g_3 + f_2g_4 - f_3g_6 \\
-f_3g_1 - f_1g_4 + f_4g_5 \\
-f_2g_1 + f_4g_2 + f_1g_6 \\
-f_3g_2 + f_1g_3 - f_2g_5
\end{bmatrix}$$

In sections 4 and 5 we will use the involution

$$\phi: \mathcal{X}^2 \to \mathcal{X}^2, \quad \phi = K \circ I^{-1} \circ \ast \circ K^{-1}$$

which we consider equivalently as an endomorphism of $C^6$. Note that its matrix coincides with the one of $\ast: \Omega^2 \to \Omega^2$.

**Proposition 2.3.** [18] The operators defined above have the following properties

(6) $\phi(u) \cdot y = u \cdot \phi(y)$, for $u, y \in C^6$
(7) $\phi(u) \cdot \phi(y) = u \cdot \phi(y)$, for $u, y \in C^6$
(8) $u \cdot (y \times z) = y \cdot (\bar{z} \times \phi(u))$, for $u \in C^6, y, z \in C^4$
(9) $(u \times z) \cdot \phi(u \times y) = 0$, for $u, y, z \in C^4$
(10) $u \bar{\otimes} (y \times z) = y \bar{\otimes} (z \times u)$, for $u, y, z \in C^4$
(11) $z \bar{\otimes} \phi(u \times y) = -(z \cdot u)y + (z \cdot y)u$, for $u, y, z \in C^4$
(12) $(u \times z) \bar{\otimes} \phi(u \times y) = -(z \cdot \phi(u \times y))u$, for $u, y, z \in C^4$
(13) $\nabla \bar{\otimes} (u \times y) = y \bar{\otimes} (\nabla \times u) - u \bar{\otimes} (\nabla \times y)$, for $u, y \in C^4$
(14) $\nabla \times Fu = \nabla F \times u + F(\nabla \times u), F \in V$, for $u \in C^4$
(15) $\nabla \times Fy = \nabla F \times y + F(\nabla \times y)$, for $F \in C, y \in C^6$
(16) $\text{Div}(Fu) = \nabla F \cdot u + F\text{Div}(u)$, for $F \in C, y \in C^4$
(17) $\text{Div}(u \bar{\otimes} y) = y \cdot \phi(\nabla \times u) - u \cdot (\nabla \times y)$, for $u \in C^4, y \in C^6$

**Proof.** Direct computation. \hfill $\square$

## 3. Poisson structure on broken Lefschetz fibrations

**Definition 3.1.** On a smooth, closed 4-manifold $X$, a broken Lefschetz fibration or blf is a smooth map $f: X \to S^2$ that is a submersion outside the singularity set. Moreover, the allowed singularities are of the following type:

1. **Lefschetz singularities:** finitely many points
   
   $$C = \{p_1, \ldots, p_k\} \subset X,$$

   which are locally modeled by complex charts

   $$\mathbb{C}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1^2 + z_2^2,$$
(2) indefinite fold singularities, also called broken, contained in the smooth embedded 1-dimensional submanifold \( \Gamma \subset X \setminus C \), and which are locally modelled by the real charts

\[ \mathbb{R}^4 \to \mathbb{R}^2, \quad (t, x_1, x_2, x_3) \mapsto (t, -x_1^2 + x_2^2 + x_3^3). \]

In [7] it was shown that a singular Poisson structure \( \pi \) of rank 2 can be associated to the fibration structure of a BLF in such a way that the fibres of \( f \) correspond to the leaves of the foliation induced by \( \pi \) and the singularity set of \( f \) is precisely the singular locus of the bivector. We recall the statement.

**Theorem 3.2.** [7] Let \( X \) be a closed oriented smooth 4-manifold. Then on each homotopy class of maps from \( X \) to the 2-sphere there exists a complete Poisson structure of rank 2 on \( X \) whose associated Poisson bivector vanishes only on a finite collection of circles and isolated points.

The local model of \( \pi \) around the singular locus \( \Gamma \) is given by

\[
\pi_{\Gamma, \mu} = \mu(t, x_1, x_2, x_3) \left( x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \right),
\]

where \( \mu \) is a non-vanishing function, and around \( C \) the local model is given by

\[
\pi_{C, \mu} = \mu(x_1, x_2, x_3, x_4) \left[ (x_3^2 + x_4^2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (-x_1 x_4 + x_2 x_3) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \right.
\]

\[
- (x_2 x_4 + x_1 x_3) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + (x_1 x_3 + x_2 x_4) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + (x_1 x_4 + x_2 x_3) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} \left. \right].
\]

where \( \mu \) is a non-vanishing function.

**4. COBOUNDARY FORMULAS FOR JACOBIAN POISSON STRUCTURES IN 4 DIMENSIONS.**

A Poisson structure on \( \mathbb{R}[x_1, \ldots, x_n] \) is called Jacobian if there are \( n - 2 \) generic Casimir functions \( P_1, \ldots, P_{n-2} \) such that the Poisson bracket of two coordinate functions is given by

\[
\{x_i, x_j\}_\mu = \mu(x_1, \ldots, x_n) \frac{dx_i \wedge dx_j \wedge dP_1 \wedge \cdots \wedge dP_{n-2}}{dx_1 \wedge \cdots \wedge dx_n}.
\]

Denote by \( \pi_\mu \) the bivector field corresponding to \( \{\cdot, \cdot\}_\mu \). It is easily checked that Jacobian structures are examples of unimodular Poisson structures; the family of isomorphisms \( \ast \) induces a family of isomorphisms

\[
H^k(\pi_\mu, \mathbb{R}^n) \cong H_{n-k}(\pi_\mu, \mathbb{R}^n),
\]

between Poisson cohomology and Poisson homology.

A basis of Hamiltonian vector fields for (20) is \( X^{\mu}_i = \langle \pi_\mu, dx_i \rangle = \mu X^1_i \) where \( X^1_i \) is simply the basis of Hamiltonian vector fields of the structure \( \{\cdot, \cdot\}_{\mu=1} \).

The next proposition contains compact general formulas for the coboundary operators in the Poisson cohomology of such structures in dimension 4.
Proposition 4.1. The coboundary operators for the formal Poisson cohomology of Jacobian Poisson structures determined by two polynomial Casimirs $P_1, P_2$ in $\mathbb{R}^4$ are

\begin{align*}
\text{(21)} & \quad d^0(g) = -\mu \nabla g \times (\nabla P_1 \times \nabla P_2) \\
\text{(22)} & \quad d^1(Y) = K^{-1} \left[ Y(\mu \nabla P_1 \times \nabla P_2) + \phi \left( d^0 \times Y \right) \right] \\
\text{(23)} & \quad d^2(W) = -D \left[ d^0 \times \phi(K(W)) + -\frac{1}{\mu} d^0(\mu) \times \phi(K(W)) + \partial(\mu \nabla P_1 \times \nabla P_2) \otimes \phi(K(W)) \right] \\
\text{(24)} & \quad d^3(Z) = \mu \left( \nabla \times D(Z) \right) \cdot \phi(\nabla P_1 \times \nabla P_2) - \frac{1}{\mu} D(Z) \cdot d^0(\mu).
\end{align*}

Proof. On the left side of these formulas, there are $k$–vector fields, while on the right side there are $k \times 1$–matrices. One should use the isomorphisms $\iota_k$ in section 2.2 to pass from one side to the other. We chose to suppress them in the proposition’s statement, however we will indicate their use in the proof.

Since $\pi_\mu = -\frac{1}{2} \sum_{i=1}^n \partial_i \wedge X_i^\mu$, it is

\begin{align*}
\text{(25)} & \quad d^0(g) = (\pi_\mu, g)_{SN} = \mu \sum_{i=1}^4 \partial_i(g) X_i^1 = \mu \sum_{i,j=1}^4 \partial_i(g) \{x_i, x_j\} \partial_j = -\mu \sum_{i=1}^4 X_i^1(g) \partial_i = -\sum_{i=1}^4 X_i^\mu(g) \partial_i.
\end{align*}

For (21), observe that since the structure is Jacobian, by (20) it is

\begin{align*}
\nabla P_1 \times \nabla P_2 = \mu^{-1} & \begin{bmatrix}
\{x_2, x_3\}_\mu \\
\{x_3, x_4\}_\mu \\
-\{x_1, x_4\}_\mu \\
\{x_1, x_2\}_\mu \\
\{x_2, x_4\}_\mu \\
-\{x_1, x_3\}_\mu
\end{bmatrix}.
\end{align*}

Then $\nabla g \times (\nabla P_1 \times \nabla P_2)$ is precisely $\sum_{i=1}^4 \mu^{-1} X_i^\mu(g) \partial_i$.

For (22), let $Y = \sum_{i=1}^4 f_i \partial_i$. Then $d^1(Y) = (\pi_\mu, Y)_{SN} = A + B$ where

\begin{align*}
A = -\sum_{i,j=1}^4 X_i^\mu(f_j) \partial_{ij}, & \quad B = \sum_{i<j=1}^4 \sum_{k=1}^4 f_k \partial_k(\{x_i, x_j\}_\mu) \partial_{ij}.
\end{align*}

Use (25), to identify the operator $d^0$ with the vector

\begin{align*}
\text{(27)} & \quad d^0 = [-X_1^\mu, -X_2^\mu, -X_3^\mu, -X_4^\mu]^T \in (C^4)^4,
\end{align*}

meaning that each entry, as a vector field, is determined by four functions on $C$ (here we restrict to formal power series). Then
\[ \iota_2(A) = - (\phi \circ K)^T (-d^0 \times \iota_2(Y)) = K^{-1}(\phi(d^0 \times \iota_2(Y))). \]

For the term \( B \), let \( Y \) act as a linear differential operator on each entry of the \( 6 \times 1 \) matrix \( \mu(\nabla P_1 \times \nabla P_2) \) given by (26). For example the first entry of the \( 6 \times 1 \) matrix \( Y(\mu(\nabla P_1 \times \nabla P_2)) = \sum_{i=1}^{4} f_i \partial_i(\{x_2, x_3\}_\mu) \). Then \( K^{-1} \) sends this matrix precisely to the part of \( d^1(Y) \) contributed by \( B \), i.e.

\[ \iota_2(B) = K^{-1}[Y(\mu(\nabla P_1 \times \nabla P_2))]. \]

For the proof of (23), let \( W = \sum_{i<j} f_{ij} \partial_{ij} \). Then \( d^2(W) = [\pi_\mu, W]_{SN} = A + B \) where after a short computation we get

\[ A = -\sum_{\text{cyl}} X^\mu_i(f_{js}) \partial_{ijs}, \quad B = \sum_{\text{cyl}} [f_{ks} \partial_k(\{x_i, x_p\}_\mu) - f_{sk} \partial_k(\{x_i, x_p\}_\mu)] \partial_{ips}. \]

Here, the sum for \( A \) is taken cyclically, for example the function contributed by \( A \) to the coefficient of \( \partial_{123} \) in \( d^2(W) \) is

\[ -X^\mu_i(f_{23}) + X^\mu_3(f_{13}) - X^\mu_4(f_{12}). \]

Also, the sum for \( B \) is taken again cyclically on \( i < p < s \). For example, to compute the function contributed by \( B \) to the coefficient of \( \partial_{123} \) we will compute the given expression for \( \partial_{ips} = \partial_{123}, \partial_{312}, \partial_{231} \). Then for e.g. \( \partial_{312} \), it is \( i = 3, p = 1, s = 2 \) and so \( f_{ks} \) is \( f_{12} \) and \( f_{sk} \) is \( f_{23}, f_{24} \). The corresponding function (contributed by the term \( \partial_{312} \) in the cyclic sum) is \( -x_{41}f_{12} - x_{23}f_{23} + x_{14}f_{24} \). The function contributed by \( B \) to the coefficient of \( \partial_{123} \) in \( d^2(W) \) is \( 2x_{41}f_{34} - x_{23}f_{23} + x_{14}f_{24} + x_{11}f_{13} + x_{22}f_{14} \).

Considering \( d^0 \) as a vector by (27), a direct computation shows that

\[ \iota_3(A) = -D[\partial_{\mu} \phi(K(\iota_3(W)))]. \]

On the other hand a somewhat long but direct computation shows that

\[ \iota_3(B) = -D[(\nabla \mu \times (\nabla P_1 \times \nabla P_2)) \times \phi(K(\iota_3(W))) + \bar{\partial}(\mu \nabla P_1 \times \nabla P_2) \bar{\Xi} \phi(K(\iota_3(W)))]]. \]

The first term in the bracket of the right hand side is understood as before: \( \iota_3 \) realizes \( W \) as a \( 6 \times 1 \) matrix, composed successively with the linear isomorphisms \( K \) and \( \phi \). The rest of the operations are determined in section 2.2. The second term stands for the following operation: The first entry of the \( 4 \times 1 \) matrix \( \bar{\partial}(\mu \nabla P_1 \times \nabla P_2) \bar{\Xi} \phi(K(\iota_3(W))) \) is the function resulting from the dot product between \( \phi(K(\iota_3(W))) \) and the \( 6 \times 1 \) matrix \( \partial_1(\mu \nabla P_1 \times \nabla P_2) \) whose entries are the \( \partial_1 \)-derivatives of the entries of \( \mu \nabla P_1 \times \nabla P_2 \). Then simplify using (21).

Finally for (24), set \( \bar{f}_k := f_{ij} \), where \( k \) is the number \( \{1, 2, 3, 4\} \setminus \{i, j, s\} \) for \( i < j < s \). We have \( d^3(Z) = [\pi_\mu, Z]_{SN} = (A + B)\partial_{1234} \), where

\[ A = -\sum_{i, k=1}^{4} (-1)^k X^\mu_i, dx_k > \partial(\bar{f}_k), \quad B = \sum_{i, j=1}^{4} (-1)^k \partial_i(\{x_i, x_k\}_\mu)\bar{f}_k. \]
Writing again \( X_i^\mu = \sum_{k=1}^{4} \{x_i, x_k\}_\mu \partial_k = \mu \sum_{k=1}^{4} \{x_i, x_k\}_1 \partial_k \), the term \( A \) is then

\[
A = -\mu \sum_{i,k=1}^{4} (-1)^k \{x_i, x_k\}_1 \partial_i (f_k) = \mu \sum_{i,k=1}^{4} (-1)^k \{x_k, x_i\}_1 \partial_i (f_k) = \sum_{k=1}^{4} (-1)^k X_k^\mu (f_k).
\]

On the other hand, a direct computation with (26) shows that

\[
(\nabla \times D(Z)) \cdot \phi(\nabla P_1 \times \nabla P_2) = \mu^{-1} \sum_{k=1}^{4} (-1)^k X_k^1 (f_k).
\]

Furthermore, since \( P_1, P_2 \) are polynomials, one can check directly that \( \iota_A (B) \) is precisely \( D(Z) \cdot (\nabla \times (\nabla P_1 \times \nabla P_2)) \). \( \square \)

5. Poisson cohomology around Lefschetz singular points.

We will choose the function \( \mu \) in the formula (19) of the model \( \pi_{C_\mu} \) to be constant. The formulas of the coboundary operators in proposition 4.1 then simplify considerably. In particular, the Casimirs of \( \pi_{C_1} \) are given by the real and imaginary parts of the parametrization of the Lefschetz points in Definition 3.1. Namely

\[
P_1 = x_1^2 - x_2^2 + x_3^2 - x_4^2, \quad P_2 = 2(x_1 x_2 + x_3 x_4).
\]

**Proposition 5.1.** For \( \mu = -\frac{1}{4} \), and \( P_1, P_2 \) as in (28), the coboundary operators of the Poisson cohomology of the model (19) are given by the following formulas

\[
d^0(g) = \frac{1}{4} \nabla g \times (\nabla P_1 \times \nabla P_2)
\]

\[
d^1(Y) = \frac{1}{4} K^{-1} \left[ \text{Div}(Y) \nabla P_1 \times \nabla P_2 + \nabla \times (Y \times \phi(\nabla P_1 \times \nabla P_2)) \right]
\]

\[
d^2(W) = \frac{1}{4} D \left[ (\nabla \times K(W)) \times \phi(\nabla P_1 \times \nabla P_2) + \nabla (K(W) \cdot \phi(\nabla P_1 \times \nabla P_2)) \right]
\]

\[
d^3(Z) = -\frac{1}{4} (\nabla \times D(Z)) \cdot \phi(\nabla P_1 \times \nabla P_2).
\]

Alternatively, \( d^3(Z) = -\frac{1}{4} \text{Div}[D(Z) \times (\nabla P_1 \times \nabla P_2)] \).

**Proof.** The formulas for \( d^0 \) and \( d^3 \) are immediate from proposition 4.1. For \( d^1 \) and \( d^2 \) a somewhat long but direct computation confirms the claim. \( \square \)

Henceforth we simplify the notation by setting \( \pi_{C_{-\frac{1}{4}}} = \pi_{C} \).

Fix \( \varpi = (\varpi_1, \varpi_2, \varpi_3, \varpi_4) \) with \( \varpi_i = 1 \) as a weight vector inducing the polynomial degree so that \( \varpi(P_i) = \varpi(P_2) = 2 \), and set also \( V = \mathbb{R}[x_1, x_2, x_3, x_4] \). With a direct check, one sees that \( P_i \) is not a zero divisor in \( V / \langle P_i \rangle \) for \( i \neq j = 1, 2 \). Hence \( P_1, P_2 \) are, by definition, a regular sequence in \( V \). Setting \( J \) to be the ideal generated by \( P_1, P_2 \) and the \( 2 \times 2 \) minors of their Jacobian matrix , we have that \( V_{\text{sing}} = V / J \).
is finite dimensional and so \((P_1, P_2)\) form a complete intersection with isolated singularity (at zero).

The Poisson homology groups have Poincaré series given in [18, Theorem 3.1], and they will have the same rank as free \(\mathbb{R}[P_1, P_2]\)-modules with the homology groups therein. Due to unimodularity of Jacobian Poisson structures, the rank of the Poisson cohomology groups is thus determined by the family of isomorphisms \(*\).

The proofs of the next propositions compute the generators of the Poisson cohomology groups \(H^k(\pi_C, B^4)\) with polynomial coefficients on a tubular neighbourhood \(U_C \approx B^4\) of a Lefschetz singularity. Since the Poisson coboundary operator \(d\) is homogeneous quadratic, one can replace \(V\) by \(V_{\text{formal}}\), the algebra of formal power series equipped with \(\pi_C\) and thus get the formal Poisson cohomology.

**Proposition 5.2.** The formal Poisson cohomology group \(H^0(\pi_C, U_C)\) is a rank 1 free \(\mathbb{R}[P_1, P_2]\)-module generated by 1.

**Proof.** Follows directly since \(\text{Rank}(\pi_C) = 2\).

**Proposition 5.3.** The formal Poisson cohomology group \(H^1(\pi_C, U_C)\) is a rank 1 free \(\mathbb{R}[P_1, P_2]\)-module generated by the Euler vector field.

**Proof.** Let
\[
\rho = \sum_{i=1}^{4} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4,
\]
and \(E = \sum_{i=1}^{4} x_i \partial_i\) be the Euler vector field. Then \(\star E = \rho\). Since \(\text{Div}(E) = 4\) is equal to \(2 \deg(P_1)\), we get the claim by [18, Theorem 3.3].

**Proposition 5.4.** The formal Poisson cohomology group \(H^4(\pi_C, U_C)\) is a free \(\mathbb{R}[P_1, P_2]\)-module of rank 7, generated by \((\nu_i)_{0 \leq i \leq 6} = (1, x_1, x_2, x_3, x_4, x_1 x_2, x_3 x_4)\).

**Proof.** Since \(d^3\) is a quadratic operator, no constant or linear terms are in \(\text{Im}(d^3)\). The kernel of \(d^4\) being \(V\), together with direct computation of the image \(d^3(X_1)\), \(X_1 \in \mathcal{X}_3^0(U_C)\), proves the claim.

**Proposition 5.5.** The formal Poisson cohomology group \(H^3(\pi_C, U_C)\) is a free \(\mathbb{R}[P_1, P_2]\)-module isomorphic to
\[
\bigoplus_{k=1}^{5} \mathbb{R}[P_1, P_2] D(\nabla \nu_k) \oplus \bigoplus_{k=0}^{5} \mathbb{R}[P_1, P_2] \nu_k D(\nabla P_2) \oplus \\
\left( \mathbb{R}[P_1, P_2] D(\nabla P_1) \right) \oplus \left( \mathbb{R}[P_1, P_2] x_1 x_2 D(\nabla P_1) \right),
\]
where \((\nu_i)_{0 \leq i \leq 6} = (1, x_1, x_2, x_3, x_4, x_1 x_2, x_3 x_4)\).

**Proof.** Let \(H \in \ker(d^3)\). Then by (32),
\[
(\nabla \times D(H)) \cdot \phi (\nabla P_1 \times \nabla P_2) = 0.
\]
By the unimodularity of Jacobian Poisson structures and [18, Prop 3.2],
\[
D(H) = \beta_1 \nabla P_1 + \beta_2 \nabla P_2 + \nabla \beta_3, \quad \beta_i \in V.
\]
Since \(D\) is an isomorphism and \(D^2 = -\text{Id}\), it is
\[
H = -D(\beta_1 \nabla P_1) - D(\beta_2 \nabla P_2) - D(\nabla \beta_3), \quad \beta_i \in V
\]
(35)
By definition it is \(H^4(\pi_C, U_C) = V/\text{Im}(d^3)\) so one can write the polynomials \(\beta_i\) as a sum of an element in \(H^4(\pi_C, U_C)\) and an element in \(\text{Im}(d^3)\). By proposition 5.4, \(H^4(\pi_C, U_C)\) is a Casimir-module with generators \(H^4(\pi_C, U_C) = \langle 1, x_1, x_2, x_3, x_4, x_1 x_2, x_3 x_4 \rangle\).
Recalling that \(H^4(\pi_C, U_C) \simeq \mathbb{R}[P_1, P_2] \otimes V_{\text{sing}}\), we write
\[
\beta_l = -\frac{1}{4} (\nabla \times D(H_l)) \cdot \phi(\nabla P_1 \times \nabla P_2) + \sum_{i=0}^{\mu_l} \sum_{j=0}^{\delta_l} \sum_{k=0}^{6} \lambda_{ijk} P_1^i P_2^j \nu_k.
\]
(36)
For \(l = 1, 2\), it is
\[
\beta_l \nabla P_l = -\frac{1}{4} (\nabla \times D(H_l)) \cdot \phi(\nabla P_1 \times \nabla P_2) \nabla P_l + \sum_{i=0}^{\mu_l} \sum_{j=0}^{\delta_l} \sum_{k=0}^{6} \lambda_{ijk} P_1^i P_2^j \nu_k \nabla P_l.
\]
(37)
We will show below that for \(l = 1, 2\), the 3-vector field
\[
B_l := -\frac{1}{4} (\nabla \times D(H_l)) \cdot \phi(\nabla P_1 \times \nabla P_2) \nabla P_l
\]
is equal to
\[
\sum_{i=0}^{\mu_l} \sum_{j=0}^{\delta_l} \sum_{k=0}^{6} \lambda_{ijk} P_1^i P_2^j \nu_k \nabla P_l.
\]
(38)

Consider \(W_l := (-1)^{l+1} K^{-1} D(H_l) \times \nabla P_l\). By (41) and (42) one then has that \(D(B_l) = d^2(W_l)\) and (39) is proved.

Thus for \(l = 1, 2\),
\[
-D(\beta_l \nabla P_l) = -d^2(W_l) - \sum_{i=0}^{\mu_l} \sum_{j=0}^{\delta_l} \sum_{k=0}^{6} \lambda_{ijk} P_1^i P_2^j D(\nu_k \nabla P_l).
\]
(43)

By (36), we have
\[
\nabla \beta_3 = -\frac{1}{4} \nabla [(\nabla \times D(H_3)) \cdot \phi(\nabla P_1 \times \nabla P_2)] + \sum_{k=0}^{6} \left( \sum_{i=0}^{\mu_3} \sum_{j=0}^{\delta_3} \lambda_{ijk} P_1^i P_2^j \nu_k \right) \nabla \nu_k.
\]
Direct computation shows that respectively
\[
\sum_{i=0}^{6} \delta_l \sum_{j=0}^{6} i \lambda_{ijk} P_1^{i-1} P_2^j \nu_k \nabla P_1 + \sum_{i=0}^{6} \delta_l \sum_{j=0}^{6} j \lambda_{ijk} P_1^i P_2^{j+1} \nu_k \nabla P_2
\]

Taking \( W_3 := K^{-1}(\nabla \times D(H_3)) \), observe that \( \nabla \times K(W_3) = 0 \) and so
\[
-\frac{1}{4} \nabla \left[ (\nabla \times D(H_3)) \cdot \phi(\nabla P_1 \times \nabla P_2) \right] = D(d^2(W_3)).
\]

So as before,
\[
-D(\nabla \beta_3) = -d^2(W_3) - \sum_{k=1}^{6} \left( \sum_{i=0}^{6} \sum_{j=0}^{6} \lambda_{ijk}^3 P_1^i P_2^j \nu_k \right) D(\nabla \nu_k)
\]
\[
- \left( \sum_{i=0}^{6} \sum_{j=0}^{6} i \lambda_{ijk}^l P_1^{i-1} P_2^j \nu_k \right) D(\nabla P_1) - \left( \sum_{i=0}^{6} \sum_{j=0}^{6} j \lambda_{ijk}^l P_1^i P_2^{j-1} \nu_k \right) D(\nabla P_2)
\]

Therefore, \( \ker(d^3) = \text{Im}(d^2) + L_3 \), where
\[
L_3 = \bigoplus_{k=1}^{6} \mathbb{R}[P_1, P_2] D(\nabla \nu_k) + \bigoplus_{k=0}^{6} \mathbb{R}[P_1, P_2] \nu_k D(\nabla P_1) + \bigoplus_{k=0}^{6} \mathbb{R}[P_1, P_2] \nu_k D(\nabla P_2).
\]

Now take \( G_1 = \phi(\nabla x_1 \times E) \) where \( E \) is the Euler vector field \( E = \sum_{i=1}^{4} x_i \partial_i \). Then
\[
(\nabla \times G_1) \times \phi(\nabla P_1 \times \nabla P_2) \quad (\text{11}) \quad = \quad \left[ (\nabla \times G_1) \cdot E - (\nabla \times E) \cdot \nabla x_1 \right] \times \phi(\nabla P_1 \times \nabla P_2)
\]
\[
= (\text{Div}(E) \cdot \nabla x_1) \times \phi(\nabla P_1 \times \nabla P_2)
\]
\[
= -4(\nabla x_1 \cdot \nabla P_2) \nabla P_2 + 4(\nabla x_1 \cdot \nabla P_2) \nabla P_1
\]
\[
(\text{11}) \quad = -8(x_1 \nabla P_2 - x_2 \nabla P_1).
\]

On the other hand,
\[
\nabla \left[ G_1 \cdot \phi(\nabla P_1 \times \nabla P_2) \right] \quad (\text{7}) \quad = \quad \nabla \left[ (\nabla x_1 \times E) \cdot (\nabla P_1 \times \nabla P_2) \right]
\]
\[
= \nabla \left[ \nabla P_1 \cdot (\nabla P_2 \times \phi(\nabla x_1 \times E)) \right]
\]
\[
= \nabla \left[ \nabla P_1 \cdot (\nabla P_2 \times (\nabla x_1 \times E) + (\nabla P_2 \cdot E) \cdot \nabla x_1) \right]
\]
\[
= \nabla \left[ \nabla P_1 \cdot (-2x_2 E + 2P_2 \nabla x_1) \right] = -4 \nabla \left( x_2 P_1 + x_1 P_2 \right),
\]
\[
\text{where we used that } \nabla P_1 \cdot E = 2P_1. \quad \text{Thus for } \overline{W}_1 := K^{-1}(G_1) \text{ we have that}
\]
\[
d^2(\overline{W}_1) = -D(x_1 \nabla P_2) + D(x_2 \nabla P_1) + D(P_1 \nabla x_2) - D(P_2 \nabla x_1).
\]

In the same way, taking \( \overline{W}_i = K^{-1}(G_i) \), \( i = 2, 3, 4 \) with \( G_i = \phi(\nabla x_i \times E) \), a direct computation shows that respectively
\[
d^2(\overline{W}_2) = D(x_1 \nabla P_1) + D(x_2 \nabla P_2) - D(P_1 \nabla x_1) - D(P_2 \nabla x_2),
\]
\[
d^2(\overline{W}_3) = D(x_4 \nabla P_1) - D(x_3 \nabla P_2) - D(P_1 \nabla x_4) + D(P_2 \nabla x_3),
\]
\[
d^2(\overline{W}_4) = D(x_3 \nabla P_1) + D(x_4 \nabla P_2) - D(P_1 \nabla x_3) - D(P_2 \nabla x_4).
\]
and so \(D(x, \nabla P_i), \ i = 1, 2, 3, 4\) are written as linear combinations of elements in

\[
\bigoplus_{k=1}^{6} \mathbb{R}[P_1, P_2]D(\nabla \nu_k), \quad \bigoplus_{k=0}^{6} \mathbb{R}[P_1, P_2]\nu_k D(\nabla P_2), \quad \text{and} \quad \text{Im}(d^2).
\]

Since \(P_2 = 2(x_1 x_2 + x_3 x_4)\), for \(Q \in \mathbb{R}[P_1, P_2]\) it is

\[
QD(\nabla x_3 x_4) = \frac{1}{2}QD(\nabla P_2) - QD(\nabla x_1 x_2), \quad \text{and}
\]

\[
Qx_3 x_4 D(\nabla P_i) = \frac{1}{2}QP_2 D(\nabla P_i) - Qx_1 x_2 D(\nabla P_i), \ i = 1, 2.
\]

Thus \(\text{Ker}(d^3) = \text{Im}(d^2) + L_1'\), where

\[
L_1' = \bigoplus_{k=1}^{5} \mathbb{R}[P_1, P_2]D(\nabla \nu_k) + \bigoplus_{k=0}^{5} \mathbb{R}[P_1, P_2]\nu_k D(\nabla P_2) + \mathbb{R}[P_1, P_2]D(\nabla P_1) + \mathbb{R}[P_1, P_2]x_1 x_2 D(\nabla P_1).
\]

\[\square\]

**Proposition 5.6.** The formal Poisson cohomology group \(H^2(\pi_C, U_C)\) is a free \(\mathbb{R}[P_1, P_2] - \)module isomorphic to

\[
\left[ \bigoplus_{k=1}^{5} \mathbb{R}[P_1, P_2]K^{-1}(\nabla \nu_k \times \nabla P_1) \right] \oplus \mathbb{R}[P_1, P_2]K^{-1}(\nabla P_1 \times \nabla P_2)
\]

where \((\nu_i)_{1 \leq i \leq 6} = (x_1, x_2, x_3, x_4, x_1 x_2, x_3 x_4)\).

**Proof.** Let \(G \in \text{Ker}(d^2)\). By the unimodularity of Jacobian Poisson structures, [18, Prop. 3.4] and the fact that \(D\) is an isomorphism, one has

\[
K(G) = \beta_0 \nabla P_1 \times \nabla P_2 + \nabla \beta_1 \times \nabla P_1 + \nabla \beta_2 \times \nabla P_2,
\]

for some \(\beta_i \in \mathbb{V}\). Let then

\[
\beta_i = -\frac{1}{4} \left( \nabla \times D(H_i) \right) \cdot \phi(\nabla P_1 \times P_2) + \sum_{i=0}^{6} \sum_{j=0}^{6} \sum_{k=0}^{6} \lambda_{ij,k} P_1^i P_2^j P_3^k \nu_k.
\]

Set \(Y_0 = H_0 \times (\nabla P_1 \times \nabla P_2)\). To compute \(d^1(Y_0)\) observe that

\[
\nabla \times \left[ Y_0 \times \phi(\nabla P_1 \times \nabla P_2) \right] \overset{(10)}{=} \nabla \times \left[ (\nabla P_1 \times (\nabla P_2 \times H_0)) \times \phi(\nabla P_1 \times \nabla P_2) \right]
\]

\[
\overset{(12)}{=} \nabla \times \left[ ((\nabla P_2 \times H_0) \cdot \phi(\nabla P_2 \times \nabla P_1)) \nabla P_1 \right]
\]

\[
\overset{(9)}{=} \phi(\nabla P_1 \times \nabla P_2) \nabla P_1 \times \nabla P_2.
\]

Furthermore it is

\[
\text{Div}\left( H_0 \times (\nabla P_1 \times \nabla P_2) \right) \nabla P_1 \times \nabla P_2
\]

\[
\overset{(17)}{=} \left[ (\nabla P_1 \times \nabla P_2) \cdot \phi(\nabla \times H_0) - H_0 \cdot \frac{\nabla \times (\nabla P_1 \times \nabla P_2)}{=0} \right] \nabla P_1 \times \nabla P_2
\]

\[
\overset{(6)}{=} (\nabla \times H_0) \cdot \phi(\nabla P_1 \times \nabla P_2) \nabla P_1 \times \nabla P_2
\]
Thus
\[ d^1(Y_0) = \frac{1}{4} K^{-1} \left[ \left( (\nabla \times H_0) \cdot \phi(\nabla P_1 \times \nabla P_2) \right) \nabla P_1 \times \nabla P_2 \right]. \]

Respectively, for \( l = 1, 2 \) and setting \( Y_l = \nabla P_l \times (\nabla \times H_l) \), we have
\[
\nabla \times [Y_l \times \phi(\nabla P_1 \times \nabla P_2)] \overset{(12)}{=} - \nabla \times \left[ \left( (\nabla \times H_l) \cdot \phi(\nabla P_1 \times \nabla P_2) \right) \nabla P_l \right] \\
\overset{(14)}{=} - \nabla \left[ (\nabla \times H_l) \cdot \phi(\nabla P_1 \times \nabla P_2) \right] \times \nabla P_l.
\]

By (17) and since \( \nabla \times \nabla P_l = 0 \) and \( \nabla \times (\nabla \times H_l) = 0 \), it is
\[
\text{Div} \left( (\nabla P_l \times (\nabla \times H_l)) \nabla P_1 \times P_2 \right) = 0.
\]

Thus
\[ d^1(Y_l) = -\frac{1}{4} K^{-1} \left[ \nabla \left( (\nabla \times H_l) \cdot \phi(\nabla P_1 \times \nabla P_2) \right) \times \nabla P_l \right]. \]

For \( Y := \lambda^l_{ijk} P^i_1 P^j_2 \nu_k E \), observe that by (16) one has
\[
\text{Div}(Y) = \lambda^l_{ijk} \left[ (2i + j + 4) P^i_1 P^j_2 \nu_k + P^i_1 P^j_2 \nu_k \cdot E \right]
\]
\[ = (2i + j + 4 + \deg(\nu_k)) \lambda^l_{ijk} P^i_1 P^j_2 \nu_k. \]

Furthermore,
\[
\nabla \times \left[ Y \times \phi(\nabla P_1 \times \nabla P_2) \right] \overset{(11)}{=} \nabla \times \left[ - (Y \cdot \nabla P_1) \nabla P_2 + (Y \cdot \nabla P_2) \nabla P_1 \right] \\
= \nabla \times \left[ - 2 \lambda^l_{ijk} P^i_1 P^j_2 \nu_k \nabla P_2 + 2 \lambda^l_{ijk} P^i_1 P^j_2 P^{i+1}_1 \nu_k \nabla P_1 \right] \\
\overset{(14)}{=} 2 \lambda^l_{ijk} \left[ - \nabla (P^i_1 P^j_2 \nu_k) \nabla P_2 + \nabla (P^i_1 P^{j+1}_2 \nu_k) \nabla P_1 \right].
\]

Then, by (30),
\[
K(4d^1(Y)) = \text{Div}(Y) \nabla P_1 \times \nabla P_2 + \nabla \times \left[ Y \times \phi(\nabla P_1 \times \nabla P_2) \right]
\]
and so by (47), (48), \( \text{Ker}(d^2) = \text{Im}(d^1) + L_2 \) where
\[
L_2 = \sum_{k=1}^{6} \mathbb{R}[P_1, P_2] K^{-1}(\nabla \nu_k \times \nabla P_1) + \sum_{k=1}^{6} \mathbb{R}[P_1, P_2] K^{-1}(\nabla \nu_k \times \nabla P_2) + \mathbb{R}[P_1, P_2] K^{-1}(\nabla P_1 \times \nabla P_2).
\]

Computing \( \nabla \times \left[ \nabla x_i \times \phi(\nabla P_1 \times \nabla P_2) \right] \) for \( i = 1, 2, 3, 4 \) we have respectively
\[
d^1(\nabla x_1) = \frac{1}{2} K^{-1} \left[ \nabla x_2 \times \nabla P_1 - \nabla x_1 \times \nabla P_2 \right] \\
d^1(\nabla x_2) = \frac{1}{2} K^{-1} \left[ \nabla x_1 \times \nabla P_1 - \nabla x_2 \times \nabla P_2 \right] \\
d^1(\nabla x_3) = \frac{1}{2} K^{-1} \left[ \nabla x_4 \times \nabla P_1 - \nabla x_3 \times \nabla P_2 \right] \\
d^1(\nabla x_4) = \frac{1}{2} K^{-1} \left[ \nabla x_3 \times \nabla P_1 - \nabla x_4 \times \nabla P_2 \right].
\]

Thus for \( i = 1, 2, 3, 4 \), \( K^{-1}(\nabla x_i \times \nabla P_2) \) is written mod \( \text{Im}(d^1) \) as linear combination of elements in \( L_2 \).

Since \( \nabla x_{12} \times \nabla P_2 = -\nabla x_{34} \times \nabla P_2 \) and \( \nabla P_1 \times \nabla P_2 = -\frac{1}{2} \nabla x_{12} \times \nabla P_1 - \frac{1}{2} \nabla x_{34} \times \nabla P_1 \)
we get the claim.
6. Poisson cohomology around singular circles

In this section we calculate the formal Poisson cohomology of the Poisson structure \( \pi_\Gamma \) around the circles of fold singularities of a bLf. We restrict to the case where the function \( \mu \) determining the conformal class is identically 1, so we work with the linear model

\[
\pi_{\Gamma_i} = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2},
\]

and drop the subscript 1 henceforth. Contrary to section 5, we will not use the simplified formulas resulting from proposition 4.1. Our computation shows that in terms of Poisson cohomology, singular circles are like isolated singularities in \( \mathbb{R}^3 \), which allows us to use the results of [19].

On the normal bundle of a singular circle there is a splitting \( \mathbb{R}^3 \times S^1 \rightarrow S^1 \) into a rank 1-bundle and a rank 2-bundle over \( S^1 \). There are two possible splittings up to isotopy [7]. One is orientable and the other non-orientable, where the former is given by the identity map and the latter is defined by the involution \( \iota : S^1 \times D^3 \rightarrow S^1 \times D^3 \)

\[
(\theta, x_1, x_2, x_3) \mapsto (\theta + \pi, -x_1, x_2, -x_3).
\]

The bivector field \( \pi_\Gamma \) is invariant under \( \iota \) and descends to the quotient of \( S^1 \times B^3 \) by the involution for the non-orientable tubular neighbourhood [7, Proposition 3.2].

Remark 6.1. The tubular neighbourhood \( U_\Gamma \simeq (S^1 \times B^3, \pi_\Gamma) \) can be regarded as the product Poisson manifold \((S^1, 0) \times (B^3, \pi_{g^*})\), where \( \pi_{g^*} \) is the Lie-Poisson structure on \( g^* = \mathfrak{sl}(2, \mathbb{R})^* \). Equivalently, we consider the Poisson manifold \((S^1 \times B^3, \pi_{g^*})\). Since \( \mathfrak{sl}(2, \mathbb{R}) \) is not compact, its Lie-Poisson cohomology \( H_\pi(\mathfrak{sl}(2, \mathbb{R})^*) \) is not equal to \( H_{\text{Lie}}(\mathfrak{sl}(2, \mathbb{R})) \otimes \mathbb{R}[Q_1, Q_2] \).

Fix the volume form \( \text{vol} = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \). A straightforward calculation shows that the modular vector field of \( \pi_\Gamma \) vanishes identically. Moreover, the extension of the Poisson structure to the regular parts is symplectic, thus the unimodularity holds everywhere and there exists a measure preserved by all Hamiltonian flows.

Let \( \pi^4 : \Omega^1(\mathbb{R}^4) \rightarrow \mathfrak{X}^1(\mathbb{R}^4) \) be again the contraction with \( \pi \), i.e., \( \pi^4(dx_i) = \langle \pi, dx_i \rangle \). The Hamiltonian vector fields of the coordinate functions are

\[
\begin{align*}
\pi^4(dx_0) &= 0, \\
\pi^4(dx_1) &= x_3 \partial_2 - x_2 \partial_3, \\
\pi^4(dx_2) &= -x_1 \partial_3 - x_3 \partial_1, \\
\pi^4(dx_3) &= x_1 \partial_2 + x_2 \partial_1.
\end{align*}
\]

For simplicity in notation we set \( X_i := \pi^4(dx_i) \).

6.1. Description of the coboundary operator. The Poisson bivector is

\[
\pi_\Gamma = -\frac{1}{2} \sum_{i=1}^{3} \partial_i \wedge X_i.
\]
For $f \in C$,

\[
d^0(f) = \sum_{i=1}^{3} \partial_i(f)X_i = -\sum_{i=1}^{3} X_i(f)\partial_i.
\]

For $Y = \sum_{i=0}^{3} f_i \partial_i \in \mathfrak{x}^1(\mathbb{R}^4)$,

\[
d^1(Y) = \sum_{i=1}^{3} X_i(f_0)\partial_{0i} + \sum_{i<j=1}^{3} \left( X_j(f_i) - X_i(f_j) + (-1)^{\lfloor \frac{i+j}{2} \rfloor} f_k \right) \partial_{ij}
\]

where $[t]$ denotes the integral part of $t \in \mathbb{R}$, for example $[3.7] = 3$ and the index $k$ is the index completing the triplet $\{i, j, k\} = \{1, 2, 3\}$ for chosen $i < j$.

Furthermore, for $W = \sum_{i<j=0}^{3} f_{ij} \partial_{ij} \in \mathfrak{x}^2(\mathbb{R}^4)$,

\[
d^2(W) = \sum_{i<j=1}^{3} \left( X_i(f_0) - X_j(f_0) + (-1)^{\lfloor \frac{i+j}{2} \rfloor} f_0k \right) \partial_{0ij} + \left( \sum_{i<j=1}^{3} (-1)^i X_i(f_{jk}) \right) \partial_{123}
\]

and finally, for $Z = \sum_{i<j<k=0}^{3} f_{ijk} \partial_{ijk} \in \mathfrak{x}^3(\mathbb{R}^4)$,

\[
d^3(Z) = \sum_{i<j=1}^{3} (-1)^{k+1} X_k(f_{0ij}) \partial_{0123}.
\]

6.2. Formal cohomology. Let again $V = \mathbb{R}[x_0, x_1, x_2, x_3]$ be the algebra of polynomials in $x_0, x_1, x_2, x_3$, $V_i = \mathbb{R}_i[x_0, x_1, x_2, x_3]$ be the vector space of homogeneous polynomials of degree $i$ and $\mathfrak{x}_i^k$ be the space of $k$-vector fields whose coefficients are elements of $V_i$. Since $\pi$ is linear, we can decompose each term $d^k : \mathfrak{x}_i^k \rightarrow \mathfrak{x}_i^{k+1}$ as $d^k = \sum_{i=0}^{\infty} \partial_i^k$ with $\partial_i^k : \mathfrak{x}_i^k \rightarrow \mathfrak{x}_i^{k+1}$.

In terms of our notation for polyvector fields and their coefficient functions the operators $d_i^k$ fit in the sequence

\[
0 \rightarrow V_i \xrightarrow{d_0^k} V_i \xrightarrow{d_1^k} V_i \xrightarrow{d_2^k} V_i \xrightarrow{d_3^k} V_i \rightarrow 0
\]

and more precisely

\[
f \xrightarrow{d_0^k} (f_0, f_1, f_2, f_3) \xrightarrow{d_1^k} (f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}) \xrightarrow{d_2^k} (f_{012}, f_{013}, f_{023}, f_{123}) \xrightarrow{d_3^k} f_{0123}.
\]

One can check using the Jacobian form of $\pi_1$ that the functions

\[
Q_1(x_0, x_1, x_2, x_3) = x_0, \text{ and } Q_2(x_0, x_1, x_2, x_3) = -x_1^2 + x_2^2 + x_3^2
\]
parametrizing the singular locus, are the generators of the algebra of Casimir functions for $\pi_\Gamma$, which henceforth is denoted by $\mathbb{R}[Q_1, Q_2]$.

**Proposition 6.2.** Let $(U_\Gamma, \pi_\Gamma)$ the tubular neighbourhood of indefinite fold singularities of a bLf with Poisson bivector as in (49). Let also $(X_{\text{formal}}^{*}(U_\Gamma), d)$ be the Poisson cochain complex of multivector fields with formal coefficients. The formal Poisson cohomology $H_{\text{formal}}^{*}(U_\Gamma, \pi_\Gamma)$ is given by the following list of free $\mathbb{R}[Q_1, Q_2]$-modules

- $H_{\text{formal}}^{0}(U_\Gamma, \pi_\Gamma) = \mathbb{R}[Q_1, Q_2]$
- $H_{\text{formal}}^{1}(U_\Gamma, \pi_\Gamma) = \mathbb{R}[Q_1, Q_2]d_0$
- $H_{\text{formal}}^{2}(U_\Gamma, \pi_\Gamma) = 0$
- $H_{\text{formal}}^{3}(U_\Gamma, \pi_\Gamma) = \mathbb{R}[Q_1, Q_2]d_{123}$
- $H_{\text{formal}}^{4}(U_\Gamma, \pi_\Gamma) = \mathbb{R}[Q_1, Q_2]d_{0123}$

**Proof.** Consider the maps

$$d_1^1 : V_i \rightarrow V_i^\otimes 3, \quad d_2^1 : V_i^\otimes 3 \rightarrow V_i^\otimes 3$$

defined by

$$f_0 \xrightarrow{d_1^1} (f_{01}, f_{02}, f_{03}) \xrightarrow{d_2^1} (f_{012}, f_{013}, f_{023}),$$

and the maps

$$d_1^2 : V_i^\otimes 3 \rightarrow V_i^\otimes 3, \quad d_2^2 : V_i^\otimes 3 \rightarrow V_i$$

defined by

$$(f_1, f_2, f_3) \xrightarrow{d_1^2} (f_{12}, f_{13}, f_{23}) \xrightarrow{d_2^2} f_{123}.$$  

One can split $d_1$ and $d_2$ as

(58) $d_1 = d_1^1 + d_2^1$

(59) $d_2 = d_1^2 + d_2^2$.

Observe that by equations (51)–(54) we have

(60) $d_2^1 = d_1^2$

and

(61) $d_1^1 = d_0$

where for the second equation we used that $X_0 = 0$.

Recall that rank($\pi_\Gamma$) = 2 and that the algebra of Casimirs is generated by $Q_1, Q_2$. Let $k_i = \dim \mathbb{R}_i[Q_1, Q_2]$ be the dimension of the space of homogeneous Casimirs of degree $i$, and $r_i = \dim V_i$.

Since $X_0 = 0$, it is $\text{Im}(d_0) \subset \ker(d_2^1)$. Using the splitting of $d_1$ and (61) one has that

$H_{\text{formal}}^{1}(U_\Gamma, \pi_\Gamma) = \ker(d_0) \oplus (\ker(d_2^1)/\text{Im}(d_0))$.

Let $\mathcal{A} = \mathbb{R}[x_1, x_2, x_3]$ and $\phi = \frac{1}{2}Q_2$. The restriction of $\pi_\Gamma$ on $\mathcal{A}$ is then determined by $\phi$ in the sense that $(x_{\sigma(i)}, x_{\sigma(j)}) = \partial_{\sigma(k)}\phi$ for every cyclic permutation $\sigma$ of $(1, 2, 3)$. Denote this Poisson algebra by $(\mathcal{A}, \pi_\phi)$. 


The $\partial_i \phi$ have only one common zero at the origin, the vertex of the cone defined by $\phi = 0$, and for this, the Milnor number of $A/\langle \partial_0 \phi, \partial_2 \phi, \partial_3 \phi \rangle$ is finite and equal to 1. Fixing the weight vector $\tilde{\omega} = (\omega_1, \omega_2, \omega_3) = (1, 1, 1)$, $\phi$ is then weight homogeneous of weight $\tilde{\omega}(\phi) = \text{deg}(\phi) = 2$ and has an isolated singularity.

From \((51),(52)\), we get

$$\ker(d^2_1) / \text{Im}(d^1) = \mathbb{R}[x_0] \otimes H^1(A, \phi),$$

where the second term on the right side is the first Poisson cohomology group of $(A, \pi_\phi)$. Let $E_0 = \sum_{r=1}^3 x_r \partial_r$ be the (weighted by $\tilde{\omega}$) Euler vector field. Since $\tilde{\omega}(\phi) \neq \text{Div}(E_0) = 3$, by [19, Proposition 4.5], it is $H^1(A, \phi) = 0$. Hence our claim for $H^1_{\text{formal}}(U_T, \pi_T)$.

By the splitting of $d^1$, $d^2$, and \((60), (61)\), it is

$$H^2_{\text{formal}}(U_T, \pi_T) = \ker(d^3_2) / \text{Im}(d^2_1) = \mathbb{R}[x_0] \otimes H^2(A, \phi).$$

Hence, by [19, Proposition 4.8] we get our claim for $H^2_{\text{formal}}(U_T, \pi_T)$.

It is easy to see checking \((54)\) directly that $\dim(\text{Im}(d^1)) = r_i - k_i$. By the result for $H^2_{\text{formal}}(U_T, \pi_T)$, one has that $\dim(\text{Im}(d^2_1)) = 3r_i$, for all $i$, which gives the claim for $H^3_{\text{formal}}(U_T, \pi_T)$. Finally, $\dim H^3(U_T, \pi_T) = k_i$ being equal to $\mathbb{R}[(Q_1, Q_2)]d_{1234}$.

Since $\pi_T$ is linear, computing the cohomology where the coefficient functions are of fixed polynomial degree $i$, determines the formal Poisson cohomology, replacing $V_i$ with $V_{\text{formal}} = \mathbb{R}[[x_0, x_1, x_2, x_3]]$. 

\[\square\]

**References**


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